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LETTER TO THE EDITOR

Non-existence of the global energy minimiser of the Abelian vortex model with sources

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Abstract. In the Abelian Higgs vortex model with a constant external source, the existence of a global energy minimiser depends on the magnitude of the external field.

It is well known that the Abelian Higgs vortex model (or the Ginzburg–Landau theory) may possess more than one gauge-distinct solution and the real physical states are believed to be given by the energy minimisers. For example, at the semi-quantum mechanical level, in order to establish the occurrence of the Meissner effect—one of the most important manifestations of superconductivity—one has to prove that the solution which shows expulsion of the magnetic field is an energy minimiser if the external field is weak, while the normal state solution will minimise the energy when the external field is sufficiently strong.

This letter is a simple observation that the global energy minimiser may not exist when the magnitude of the external source is beyond a critical value.

The free-energy density of the static Abelian vortex model with a Higgs meson is given by

$$\mathscr{E}(A,\phi)=\tfrac{1}{4}F_{jk}^2+\tfrac{1}{2}D_j\phi(D_j\phi)^*+\tfrac{1}{8}\lambda(|\phi|^2-1)^2$$

where $\lambda>0$ is a dimensionless coupling constant with $\lambda<1$ and $\lambda>1$ describing type-I and -II superconductivity respectively, $F_{jk}=\partial_jA_k-\partial_kA_j$ is the magnetic field, ϕ is the wavefunction of the Cooper pairs and $D_j\phi=\partial_j\phi-\mathrm{i}A_j\phi$, $j,k=1,2$.

Under the influence of an external magnetic field F_{jk}^{ex}, the total energy is [1]

$$E(A,\phi)=\int_{\mathbb{R}^2}\mathscr{E}\,\mathrm{d}^2x-\tfrac{1}{2}\int_{\mathbb{R}^2}F_{jk}F_{jk}^{\mathrm{ex}}\,\mathrm{d}^2x.$$

If F_{jk}^{ex} is a constant field, the Euler–Lagrange equations of the functional E are the same as the sourceless Ginzburg–Landau equations; hence, according to [2], finite (free) energy solutions exhibit exponential vacuum decay properties and the total excited flux is quantised:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}F_{12}\,\mathrm{d}^2x=N \tag{1}$$

where N is an integer which is also recognised as the winding number of the wavefunction ϕ on the circle at infinity of $\mathbb{R}^2$.

Hence, following Bogomol'nyi [3] one has, by virtue of (1) and an integration by parts,

$$E(A,\phi)=\pi(|N|-2NF_{12}^{\mathrm{ex}}) + \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{d}^2x \left\{ |F_{12}\pm\tfrac{1}{2}(|\phi|^2-1)|^2 + |D_1\phi\pm \mathrm{i}D_2\phi|^2 + \frac{\lambda-1}{4}(|\phi|^2-1)^2 \right\} \tag{2}$$

for $N=\pm|N|$.

Now we assume $\lambda\geqslant 1$.

If the external field is so weak that

$$|F_{12}^{\mathrm{ex}}|\leqslant\tfrac{1}{2} \tag{3}$$

then $E(A,\phi)\geqslant 0$ and the global minimum is saturated by the superconducting vacua $A=0$, $|\phi|=1$. This partially proves the Meissner effect for type-II superconducting materials. However, if (3) is violated, more complicated issues appear.

As an illustration, let us make the critical choice $\lambda=1$.

In this case the Ginzburg–Landau equations and the Bogomol'nyi equations

$$F_{12}\pm\tfrac{1}{2}(|\phi|^2-1)=0$$

$$D_1\phi\pm \mathrm{i}D_2\phi=0$$

are equivalent [2] and, from (2),

$$E(A,\phi)=\pi(|N|-2NF_{12}^{\mathrm{ex}}) \tag{4}$$

if (A,ϕ) is a solution of the model.

The simple formula (4) has interesting implications.

For $|F_{12}^{\mathrm{ex}}|<\frac{1}{2}$, the superconducting vacua are the only energy minimisers which show that weak external fields cannot penetrate the superconductor as expected.

For $|F_{12}^{\mathrm{ex}}|=\frac{1}{2}$, the energy minima are attained at the vacuum solutions as well as at the N-vortex solutions provided sgn $N=$ sgn F_{12}^{ex}. This describes a 'pattern selection' phenomenon: the orientation of the vortices depends on the direction of the external field.

Finally, by virtue of (4), if $|F_{12}^{\mathrm{ex}}|>\frac{1}{2}$, arbitrarily low energy levels can be occupied by finite-energy solutions with large vortex numbers N satisfying sgn $N=$ sgn F_{12}^{ex}. This situation seems to be delicate.

References

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